Option Pricing

Quantitative Finance

2022-2023

TÉCNICO LISBOA

Outline

1. Risk (definition and propoerties)
2. Options
3. Options strategies
4. **Put-call parity**
5. **Options pricing**

Rewind

Option strategies

Payoff and profits

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Rewind

A two-year European call option at 90 Euros strike is quoted at 5 Euros and a two-year European call option at 95 Euros strike is quoted at 5.1 Euros. Design a trading strategy to lock profits (as the two prices allow arbitrage!).

Pricing Options

Upper bounds on prices $c \leq S(0)$ $[C \leq S(0)]$

Imagine that $c > S(0)$ we show that this leeds to an arbitrage portfolio, and therefore if means that the call option is overpriced, therefore I'm going to sell it:

$(t=0)$
- Sell this option and receive ,c
- Buy the underlying stock

$(c - S(0))$

$\downarrow$

Pricing Options

Upper bounds on prices $c \leq S(0)$ $[C \leq S(0)]$

$t=0$

- Sell this option and receive c
- Buy the underlying stock

$(c - S(0))$

$t=T$

- $S(T) < K$: The option is not exercised
  Profit: $(c - S(0))e^{RT} + S(T) > 0$
- $S(T) > K$: The option is exercised
  Profit: $(c - S(0))e^{RT} + K > 0$

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Pricing Options

Lower bounds on prices $c \geqslant S(0) - Ke^{-RT}$

Consider the folowing where: c=3, S=20; K=18, R=10% (per year); T=1 year.

So we show that this leads to na arbitrage possibility.

$$S(0) - Ke^{-RT} = 3{,}71$$

with $c = 3 < 3{,}71 \Rightarrow \boxed{c < S(0) - Ke^{-RT}}$

Means that the option is underpriced then you want to buy it

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Lower bounds on prices $c \geq S(0) - Ke^{-RT}$

$t=0$

- Short-Sell the stock and receive $S_{(0)}=20$
- Using that money you buy the call option

  You still have: $S(0) - c = 20 - 3 = 17$

  ↳ INVEST IN THE BANK

$t=T$

- $S(T) < 18$: You do not exercise the option

  Profit: $18,79 - S(T) > 0$

- $S(T) > 18$: The option is exercised

  Profit: $18.79 - 18 = 0,79 > 0$

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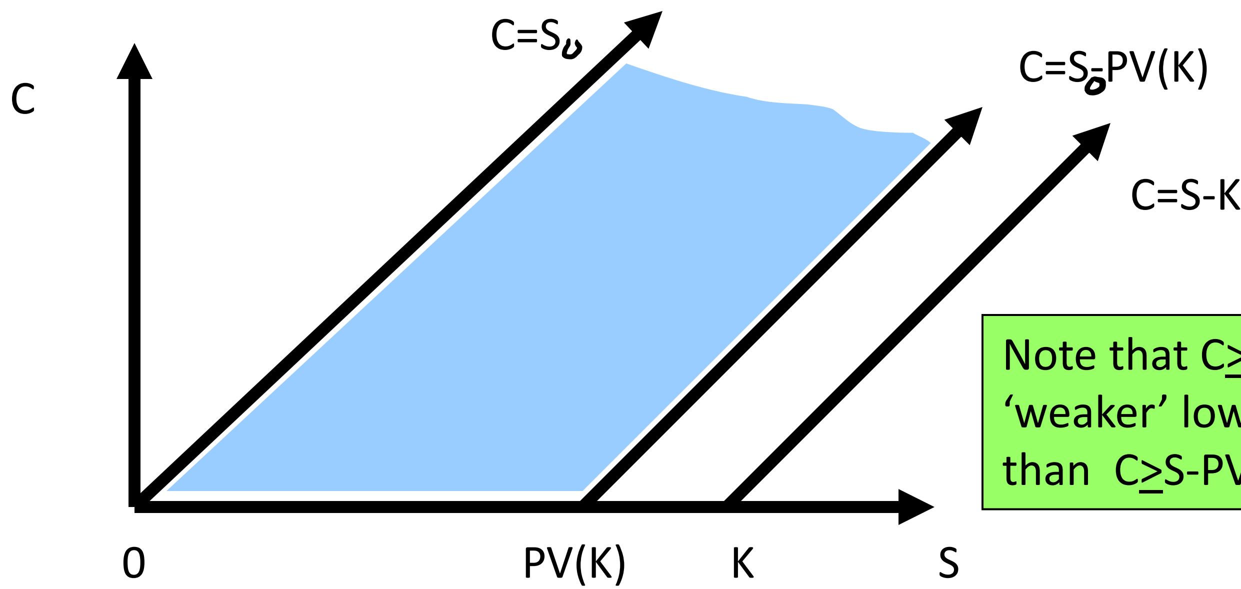
Pricing Options

Calls

The upper bound is $C \le S$

If there are no dividend payouts, the lower bound is $C \ge \max[0, S - Ke^{-rT}]$; i.e., $C \ge \max[0, S - PV(K)]$



Note that $C \geq S-K$ is a 'weaker' lower bound than $C \geq S-PV(K)$

Pricing Options

European Put Lower Bound

$p \geq Ke^{-RT} - S(0)$

What if: $p < Ke^{-rT} - S$?

Then, $p - Ke^{-rT} + S(0) < 0$

Or, $-p + Ke^{-rT} - S(0) > 0$

Today		At expiration: $S_T > K$	$S_T < K$
Buy put	$-p$	0	$+(K-S_T)$
Borrow	$+Ke^{-rT}$	$-K$	$-K$
Buy stock	$-S$	$+S_T$	$+S_T$
	>0	≥ 0	0

So, if $P < Ke^{-rT} - S(0)$, an arbitrage is possible, because the trader can receive a cash in-flow today, and not have to pay money in the future (in fact, in some cases, the trader receives money in the future, too.

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Pricing Options

Put-call partity

Consider the following 2 portfolios:

Portfolio A: you own a call option and you have $\underline{Ke^{-RT}}$

Portfolio B: you own a put option and a share $S(0)$

at time T:

Value of portfolio A= $\max(S(t)-K,0)+K = \max(S(t),K)$

Value of portfolio B= $\max(K-S(t),0)+S(t) = \max(K,S(t))$

So, as value (portfolio A) = (porfolio B) whatever happens, then they initial prices (by the non-dominance principle) have to be the same

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Pricing Options

		$S_T > K$	$S_T < K$
Portfolio A	Call option	$S_T - K$	0
	Zero-coupon bond	K	K
	Total	S_T	K
Portfolio C	Put Option	0	$K - S_T$
	Share	S_T	S_T
	Total	S_T	K

Both are worth max(S_T, K) at the maturity of the options, they must therefore be worth the same today. This means that:

$$c + Ke^{-rT} = p + S_{(0)}$$

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Pricing Options

Put-call partity

Call option: $g(t) = (S(T) - K)^+$

Put option: $g(t) = (K - S(t))^+$

$$c + Ke^{-RT} = p + S(0)$$

Total amount of money in order to buy a call option and be able to exercise it

Total amount of money in order to buy a put option and be able to exercise it

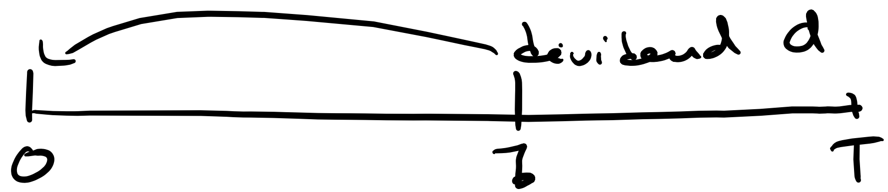
Put-Call parity is quite relevant, and in fact tell us that we only need to price "half" of the contracts.

Pricing Options

Put-call partity

We are assuming that there are no dividends distributed to the stockholders.

In case you know that you will receive dividends, with present value equals to D



$D = d e^{-R\tau}$

The put-call parity is changed, in order to take into account the dividend:

$$c + Ke^{-rT} = p + S_{(0)} - D$$

Pricing Options

Put-call partity

Suppose that

$c = 3$ $S_0 = 31$

$T = 0.25$ $r = 10\%$

$K = 30$ $D = 0$

What are the arbitrage possibilities when

$p = 2.25$?

$p = 1$?

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Put-call partity

What if: $c-p>S-Ke^{-rT}$?

Then: $c-p-S+Ke^{-rT} > 0$

		At Expiration:	
Today:		$S_T<K$	$S_T>K$
Sell call	$+c$	0	$-(S_T-K)$
Buy put	$-p$	$+(K-S_T)$	0
Buy stock	$-S$	$+S_T$	$+S_T$
Borrow	$+Ke^{-rT}$	$-K$	$-K$
	>0	0	0

Therefore, if $c-p>S-K^{-rT}$, an arbitrage is possible, because the trader receives a cash inflow today, but does not have a cash out-flow in the future.

Put-call partity

Suppose c = 4.50, p = 2.50, S = 42, K = 40, r = 6%, T = 3 month, $Ke^{-rT} = 39.40$
C–P = 4.5–2.5 = 2 < S–Ke^{-rT}= 42 - 39.40 = 2.60.

		At Expiration:	
Today:		S_T=37	S_T=44
Buy call	-4.50	0	+4
Sell put	+2.50	- 3	0
Sell stock	+42.00	-37	-44
Lend	-39.40	+40	+40
	+0.60	0	0

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Put-call partity

a) A European call option and an European put option on a stock that expires in one year have both a strike price of 44 Euros. The current stock price is 40 Euros and the one-year risk free interest rate is 10%. The price of the call is 10 Euros and the price of the put is 7 Euros. Design an arbitrage possibility

b) The present price of a stock is 50. The price of a European call option with strike price 47.5 and maturity 180 days is 4.375. The cost of a risk-free 1 euro bond 180 days is 0.98. a) Consider a European put option with price 1.025 (same strike price and maturity as the call option). Show that this is inconsistent with put-call parity. b) Describe how you can take advantage of this situation, creating an arbitrage possibility example

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Pricing Options

Put-call partity

- Rearrange, the basic put-call parity proposition to be $-c=-S+Ke^{-rT}-p$. This says that buying a call is like borrowing to buy stock; i.e., it is like buying stock on margin. But in addition, the call owner also owns a put, providing downside protection.
- If r > 0, an at the money call is worth more than an at the money put with the same K and T.
- Given S, r, and T, then c-p is known, regardless of the bullishness or bearishness that may pervade the market.
- You can replicate the payoff from any position with the other three securities (e.g. buying a put = selling stock, lending, and buying a call).

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Put-call partity

$S = c - p + Ke^{-rT}$	Sell stock short = write call, buy put, & borrow
$-S = -c + p - Ke^{-rT}$	Buy stock = buy call, write put & lend
$-p = S - c - Ke^{-rT}$	Buy put = sell stock short, buy call, & lend
$p = -S + c + Ke^{-rT}$	Write put = buy stock, write call, & borrow
$c = S + p - Ke^{-rT}$	Write call = sell stock short, write put, & lend
$-c = -S - p + Ke^{-rT}$	Buy call = buy stock, buy put, & borrow
$Ke^{-rT} = S - c + p$	Riskless borrowing = sell stock short, buy call, & write put
$-Ke^{-rT} = -S + c - p$	Riskless lending = buy stock, write call, & buy put

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Pricing Options

Main Ideas:

- A financial derivative is **defined in terms of** some underlying asset which already exists on the market.
- The derivative cannot therefore be priced arbitrarily **in relation to the underlying prices** if we want to **avoid mispricing between the derivative and the underlying price**.
- We thus want to price the derivative in a way that is **consistent** with the underlying prices given by the market.
- We are **not** trying to compute the price of the derivative in some "absolute" sense. The idea instead is to determine the price of the derivative **in terms of the market prices of the underlying assets**.

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The Binomial model

The model is very easy to understand, almost all important concepts which we will study later on already appear in the binomial case, the mathematics required to analyse it is easy, and last but not least the binomial model is often used in practice.

Assumptions of the Binomial Model

- There are two (and only two) possible prices for the underlying asset on the next date. The underlying price will either:

  Increase by a factor of u% (an uptick)
  Decrease by a factor of d% (a downtick)

- The uncertainty is that we do not know which of the two prices will be realized.

- No dividends.

- The one-period interest rate, r, is constant over the life of the option (r% per period).

- Markets are perfect (no commissions, bid-ask spreads, taxes, price pressure, etc.)

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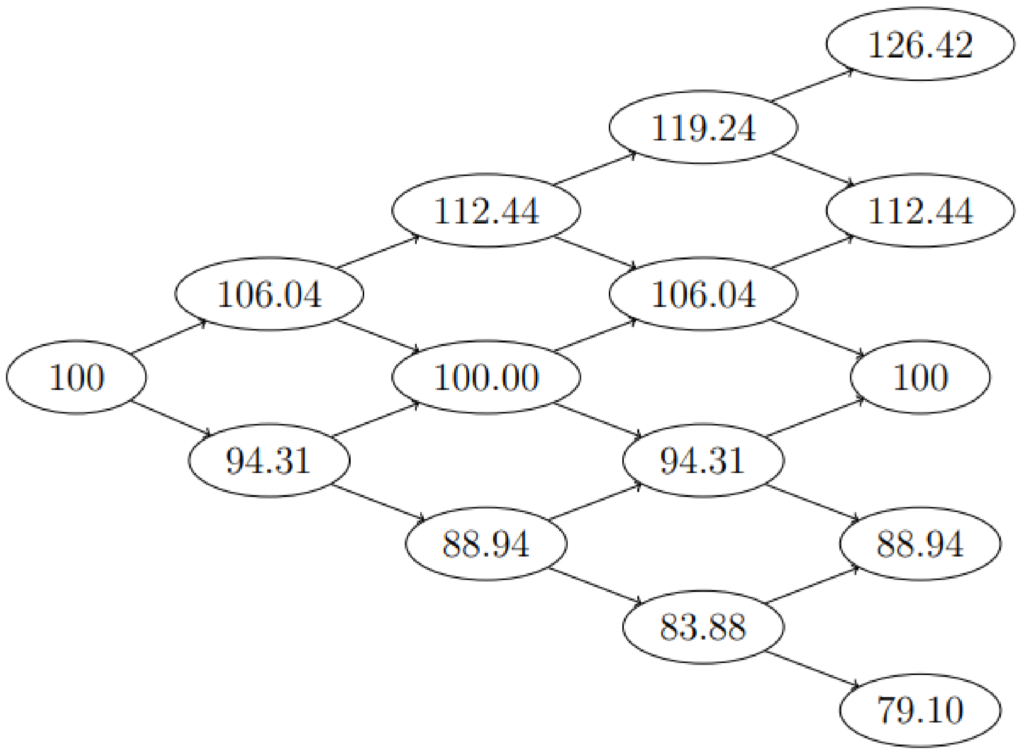
Consider a European call option, with an exercise price of 100 euro, with a maturity of 4 months. The current price of the underlying asset is 100 euro, the interest rate is 0 % and the asset may rise 6.04 % or fall 5.69 %. A binomial model is considered, with a time interval of one month. In this case, the price of the underlying asset takes the following possible values:



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The Binomial Option Pricing Model

We begin with a single period.

Then, we stitch single periods together to form the Multi-Period Binomial Option Pricing Model.

The Multi-Period Binomial Option Pricing Model is extremely flexible, hence valuable; it can value American options (which can be exercised early).

Pricing Options

Binomial Option-Pricing Model of Cox, Ross, and Rubinstein (1979):

- Consider One-Period Call Option On Stock XYZ
- Current Stock Price S_0
- Strike Price K
- Option Expires Tomorrow, $C_1 = \text{Max}\ [S_1 - K\ ,\ 0]$
- What Is Today's Option Price C_0?

0 — 1

$S_0\ ,\ C_0 = ???$

$S_1\ ,\ C_1 = \text{Max}\ [S_1 - K\ ,\ 0]$

Pricing Options

Suppose $S_0 \rightarrow S_1$ Is A Bernoulli Trial:

$$
\begin{tikzcd}[column sep=large]
 & uS_0 \\
S_1 \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\
 & dS_0
\end{tikzcd}
\qquad u > d
$$

$$
\begin{tikzcd}[column sep=large]
 & \mathrm{Max}\,[\, uS_0 - K \,,\, 0 \,] == C_u \\
C_1 \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\
 & \mathrm{Max}\,[\, dS_0 - K \,,\, 0 \,] == C_d
\end{tikzcd}
$$

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Pricing Options

Now What Should $C_0 = f(\cdot)$ Depend On?

- Parameters: S_0 , K , u , d , p , r

Consider Portfolio of Stocks and Bonds Today:

- Δ Shares of Stock, \$$B$ of Bonds
- Total Cost Today: $V_0 == S_0\Delta + B$
- Payoff V_1 Tomorrow:

$$V_1 \begin{cases} \xrightarrow{\;p\;} & uS_0\Delta + r\mathrm{B} \\ \xrightarrow{\;1-p\;} & dS_0\Delta + r\mathrm{B} \end{cases}$$

$r \equiv (1+r)$

Pricing Options

Now Choose Δ and *B* So That:

$$V_1 \begin{cases} \xrightarrow{p} & uS_0\Delta + rB = C_u \\ \xrightarrow{1-p} & dS_0\Delta + rB = C_d \end{cases} \Rightarrow \begin{aligned} \Delta^* &= (C_u - C_d)/(u-d)S_0 \\ B^* &= (uC_d - dC_u)/(u-d)r \end{aligned}$$

Then It Must Follow That:

$$C_0 = V_0 = S_0\,\Delta^* + B^* = \frac{1}{r}\left(\frac{r-d}{u-d}\,C_u + \frac{u-r}{u-d}\,C_d\right)$$

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Pricing Options

Suppose $C_0 > V_0$

- Today: Sell C_0 , Buy V_0 , Receive $C_0 - V_0 > 0$
- Tomorrow: Owe C_1, But V_1 Is Equal To C_1!

Suppose $C_0 < V_0$

- Today: Buy C_0 , Sell V_0 , Receive $V_0 - C_0 > 0$
- Tomorrow: Owe V_1, But C_1 Is Equal To V_1!

$C_0 = V_0$, Otherwise Arbitrage Exists

$$\boxed{C_0 = V_0 = S_0\,\Delta^* + B^* = \frac{1}{r}\left(\frac{r-d}{u-d}\,C_u + \frac{u-r}{u-d}\,C_d \right)}$$

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Pricing Options

Note That *p* Does Not Appear Anywhere!

- Can disagree on *p*, but must agree on option price
- If price violates this relation, arbitrage!
- A multi-period generalization exists:

$$C_0 = \frac{1}{r^n}\sum_{k=0}^{n}\binom{n}{k} p^{*k}(1-p^*)^{n-k}\mathrm{Max}[0, u^k d^{n-k} S_0 - K]$$

$$p^* \equiv \frac{r-d}{u-d}$$

- Continuous-time/continuous-state version (Black-Scholes/Merton):

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} = rC$$

$$C(S,T) = \mathrm{Max}\,[\,S-K, 0\,]$$
$$C(0,t) = 0$$

Pricing Options

Example

$S_0 = 40$, $u = 25\%$, $d = -10\%$, $r = 5\%$, $K = 45$,

What are the values of Δ, B, and C_0?

What if $C_0 = 3$?

What if $C_0 = 1$?

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Pricing Options

Let's go for another model description:

Running time is denoted by the letter t, and by definition we have two points in time, $t = 0$ ("today") and $t = 1$ ("tomorrow"). In the model we have two assets: a **bond** and a **stock**. At time t the price of a bond is denoted by B_t , and the price of one share of the stock is denoted by S_t . Thus we have two price processes B and S.

The bond price process is deterministic and given by: $B_0 = 1$,

$B_1 = 1 + r$.

The constant R is the spot rate for the period, and we can also interpret the existence of the bond as the existence of a bank with R as its rate of interest.

The stock price process is a stochastic process, and its dynamical behaviour is described as follows: $S_0 = s$,

$$S_1 \begin{cases} S_u & \text{with probability } p_u \\ S_d & \text{with probability } p_d \end{cases}$$

The Option Pricing 'Process'

Portfolios and arbitrage

We will study the behaviour of various portfolios on the (B,S) market, and to this end we define a portfolio as a vector *h = (x,y).* The interpretation is that x is the number of bonds we hold in our portfolio, whereas y is the number of units of the stock held by us.

Now let *h* be a portfolio, assume that at time t=0, this portfolio is composed by y_0 Shares of stock, x_0 of Bonds where $x_0 = - y_0 S_0$

- $h_0 = (- y_0 S_{0,} y_0) \Rightarrow V_0^h = 0$

Now we see how the portfolio evolves.

At time T=1

$$h_0(-y_0 S_0, y_0) \begin{cases} \nearrow (-y_0 S_0(1+r), y_0 S_0 u) \\ \searrow (-y_0 S_0(1+r), y_0 S_0 d) \end{cases}$$

The Option Pricing 'Process'

$$V_1^h = \begin{cases} -y_0 S_0 (1+r) + y_0 S_0 u \\ -y_0 S_0 (1+r) + y_0 S_0 d \end{cases}$$

$$= S_0 y_0 \begin{cases} -(1+r) + u > 0 \\ -(1+r) + d < 0 \end{cases}$$

If $y_0 > 0$ then the value of the portfolio V_1^h is positive with probability p_u and it is negative with probability p_d, meaning that this is not an arbitrage portfolio.

The Option Pricing 'Process'

If $y_0 < 0$ then …

So far the non-arbitrage condition involves only *u* and *d*. So what is the role of p_u and p_d ? We will see that the result of the the price of the contingent claim do not depend on p_u and p_d .

Some consequences of: $0 < d < 1 + r < u$

It follows that we may write 1+ r as: $1 + r = q_u \cdot u + q_d \cdot d,$

where q_u , q_d ≥ 0 and $q_u + q_d = 1$

In particular we see that the weights q_u and q_d can be interpreted as probabilities for a new probability measure Q

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So : $1+r = q \cdot u + (1-q)d$ with $q \in (0,1)$

this MEAN THAT $Q = (q, 1-q)$, this is PROBABILITY FUNCTION AND THIS is UNIQUE :

$$\begin{cases} 1+r = q_u u + q_d d \\ q_u + q_d = 1 \end{cases} \Leftrightarrow \begin{cases} q_u = \dfrac{(1+r)-d}{u-d} \\ q_d = \dfrac{u-(1+r)}{u-d} \end{cases}$$

$$Q = \left(\frac{1+r-d}{u-d} ; \frac{u-(1+r)}{u-d} \right) \Rightarrow$$ MARTINGALE PROBABILITY

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Due to the definition of Q, it folows that:

S_0 is given

S_1 is a random variable

$= S_0 Z$ with $Z \begin{cases} u \\ d \end{cases}$

$$E^P[S_1] = p_u S_0 u + p_d S_0 d \qquad P = (p_u, p_d)$$

$$E^Q[S_1] = q_u S_0 u + q_d S_0 d \qquad Q = \left(\frac{1+r-d}{u-d}; \frac{u-(1+r)}{u-d}\right)$$

$$= \frac{1+r-d}{u-d} S_0 u + \frac{u-(1+r)}{u-d} S_0 d$$

$$= \frac{S_0}{u-d}(1+r)(u-d) = S_0(1+r)$$

Pricing Options

$$\boxed{S_0 = \frac{1}{1+r} E^Q[S_1]} \Leftrightarrow E^Q[S_1] = (1+r)S_0$$

It means that under Q (martingale probability) today's stock price as the discounted expected value of tomorrow's stock price.

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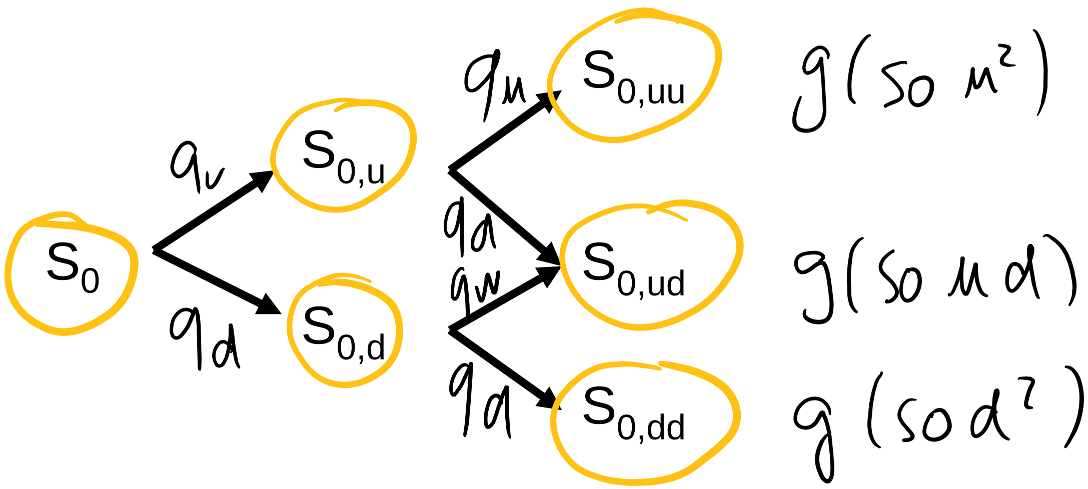
Price of contingent claims

Imagine that $x=g(S_T)$ is an option, with contract function g



$Q = (q_u, q_d)$: MARTINGALE PROBABILITY

PAYOFF OF X, WICH DEPENDS ON THE VALUE OF St

Price of contingent claims

How do we price this contingent claim?

By two ways:

- By replicating portfolio
- By risk neutral measure (Q)

<u>Replicating Portfolio</u>

We say that h is a replicating portfolio of x (derivative, contingent claim, the option) if:

$V_t^h = X$ with PROBABILITY ONE

So how do we build this replicating portfolio ?

Price of contingent claims

How do we price this contingent claim?

$$\begin{tikzcd} & S_{0,u} & g(S_0 u) \\ S_0 \arrow[ur, "*"] \arrow[dr, "**"'] & & \\ & S_{0,d} & g(S_0 d) \end{tikzcd}$$

$$\underline{\qquad\qquad}$$
$$X$$

$h = \{(x_t, y_t), t \in \{0,1\}\}$ such that

$V_1^h = g(S_1)$, i.e: $h_0 = (x, y)$

then:

$$* \quad (1+r)x + y u S_0 = g(S_0 u)$$

$$** \quad (1+r)x + y \cdot d S_0 = g(S_0 d)$$

Solving for x, y

$$\begin{cases} (1+r)x + y u S_0 = g(S_0 u) \\ (1+r)x + y d S_0 = g(S_0 d) \end{cases} \Leftrightarrow \begin{cases} x = \dfrac{1}{1+r} \dfrac{u g(S_0 d) - d g(S_0 u)}{u-d} \\ y = \dfrac{1}{S_0} \dfrac{g(S_0 u) - g(S_0 d)}{u-d} \end{cases}$$

$$\downarrow$$

Price of contingent claims

Because $d<1+r<u$ this system as one and only one solution

ie, you start with

$$x = \frac{1}{1+r} \frac{u\,g(s_0 d) - d\,g(s_0 u)}{u-d} \text{ in the bank}$$

and with

$$y = \frac{1}{s_0} \frac{g(s_0 u) - g(s_0 d)}{u-d} \text{ stocks}$$

Wich MEANS THAT YOUR INITIAL INVESTMENT IS:

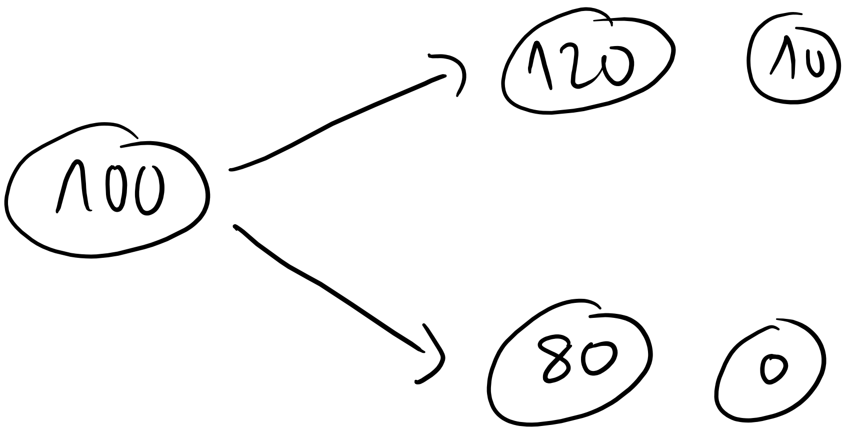
$$x + y s_0$$

then

$$\Pi(0;X) = V_0^h = x + y s_0$$

Price of contingent claims

Example $\quad u = 1{,}2 \quad d = 0{,}8 \quad r = 0 \quad$ call option $K = 110$

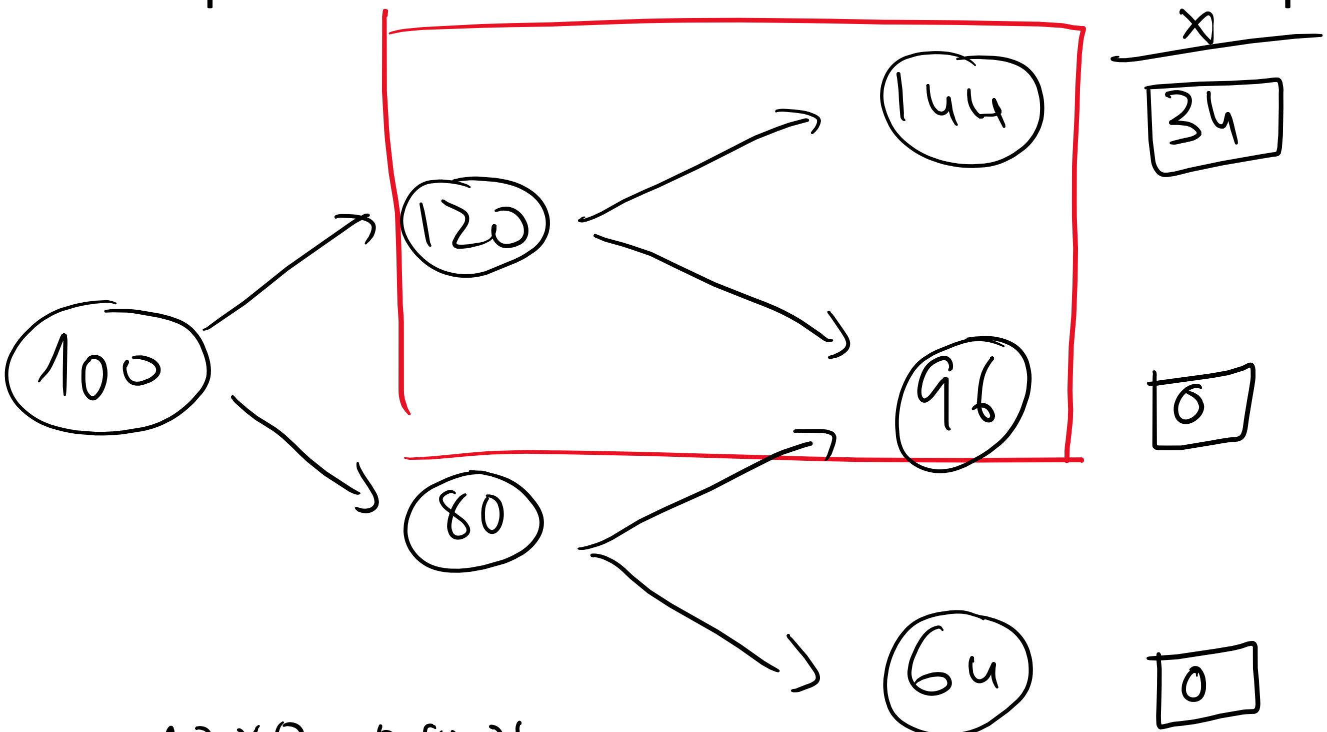


$$x = \frac{1}{1+0} \; \frac{1{,}2 \times 0 - 0{,}8 \times 10}{1{,}2 - 0{,}8} = -20$$

$$y = \frac{1}{100} \; \frac{10-0}{1{,}2 - 0{,}8} = 0{,}25$$

Price of contingent claims

How to price contracts with more then 1 time-step?



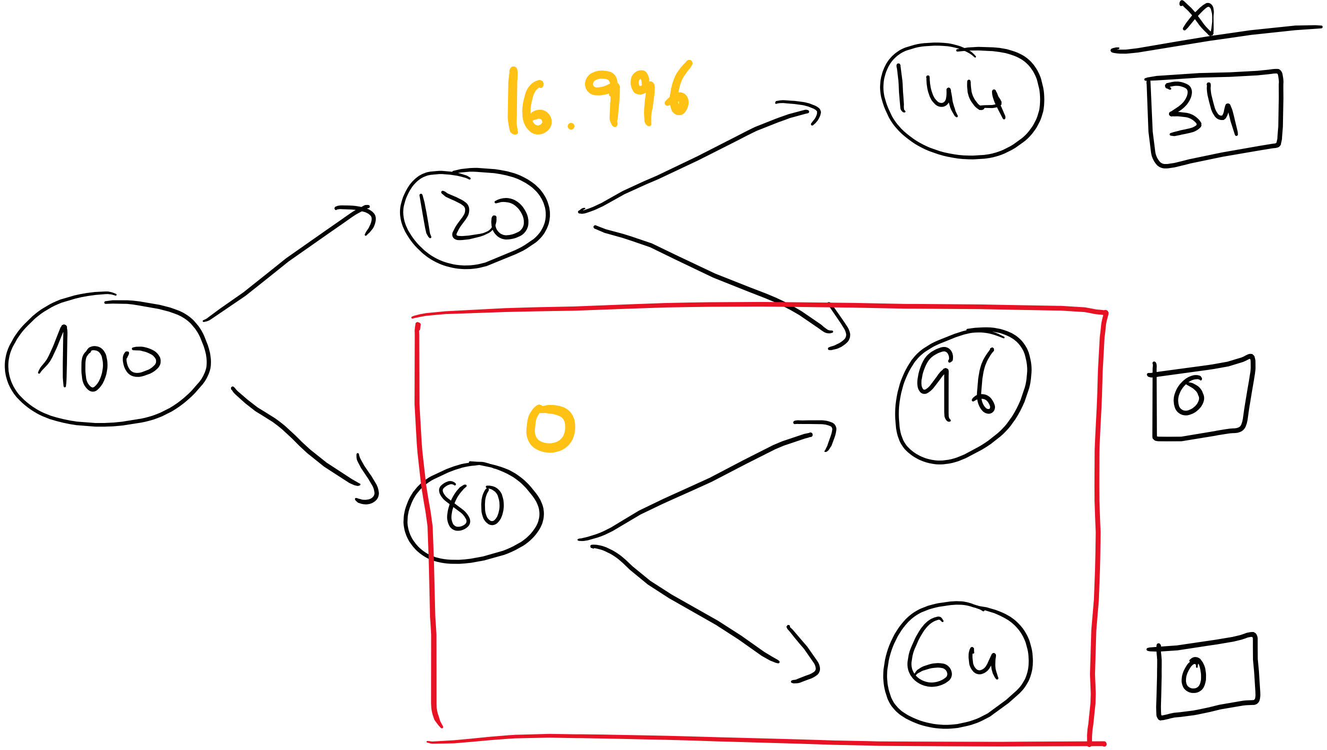
$$x_1 = \frac{1{,}2 \times 0 - 0{,}8 \times 34}{0{,}4} = -68$$

$$y_1 = \frac{1}{120} \, \frac{34-0}{0{,}4} = 0{,}7083$$

value $= 16{,}996$

Price of contingent claims

How to price contracts with more then 1 time-step?

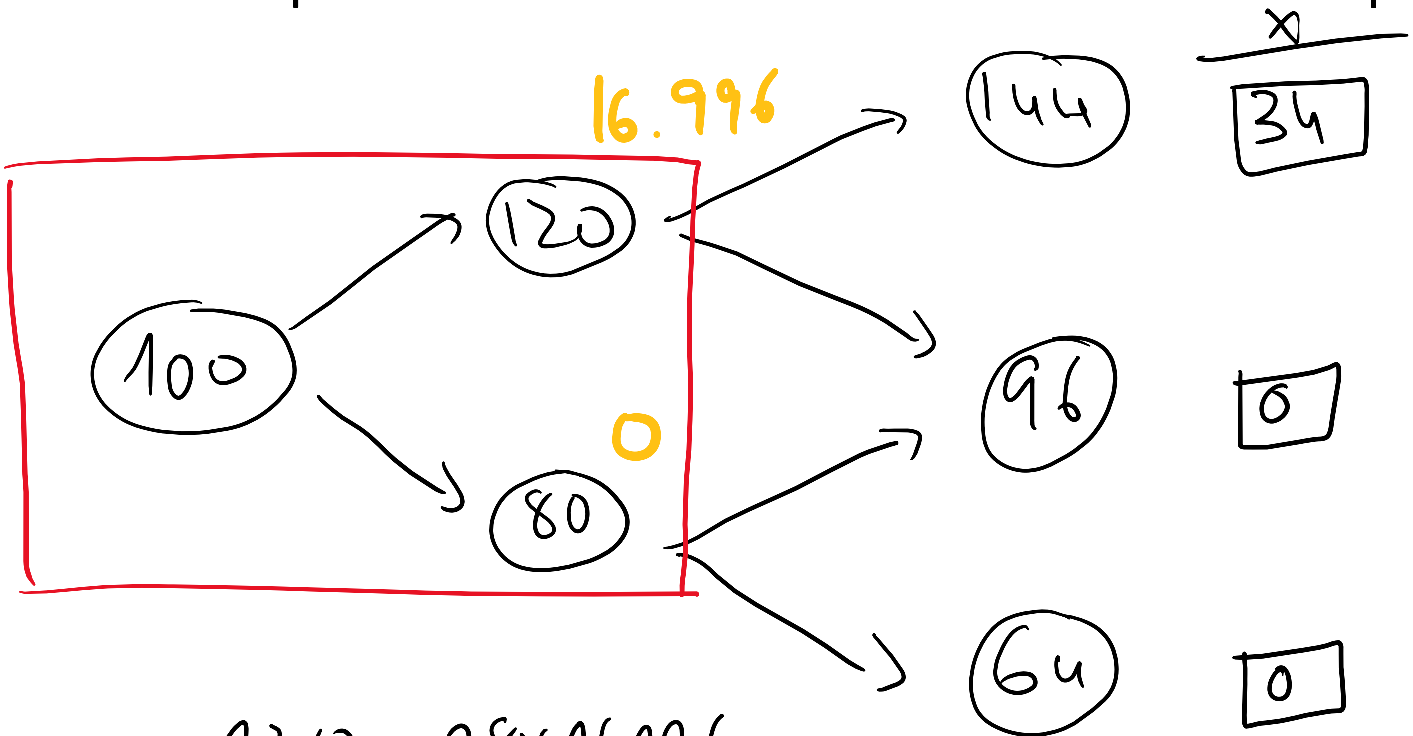


$x_2 = 0$

$y_2 = 0$ } VALUE $= 0$

Price of contingent claims

How to price contracts with more then 1 time-step?



$$x_3 = \frac{1{,}2 \times 0 - 0{,}8 \times 16{,}996}{0{,}4} = -33{,}992$$

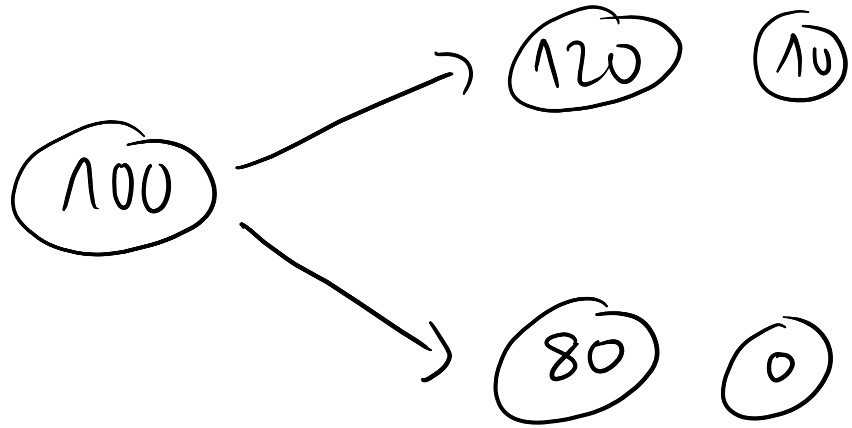
$$y_3 = \frac{1}{100} \frac{16{,}996 - 0}{0{,}4} = 0{,}4249$$

$$-33{,}993 + 0{,}4249 \times 100 \boxed{= 8{,}5}$$

Price of contingent claims

<u>Risk Neutral Measure</u>

Returning to the example: $u = 1{,}2 \quad d = 0{,}8 \quad r = 0$



It is possible to compute the martingale probabilities

$$Q = \left(\frac{1+r-d}{u-d} \,;\, \frac{u-(1+r)}{u-d} \right) = \left(\frac{1-0{,}8}{0{,}4} = 0{,}5 \,;\, 0{,}5 \right)$$

USING THE FORMULA $E^{Q}[S_1] = (1+r)S_0$

Price of contingent claims

$$E^{Q}[X] = E^{Q}[g(S_1)] = g(S_0u)q_u + g(S_0d)q_d$$

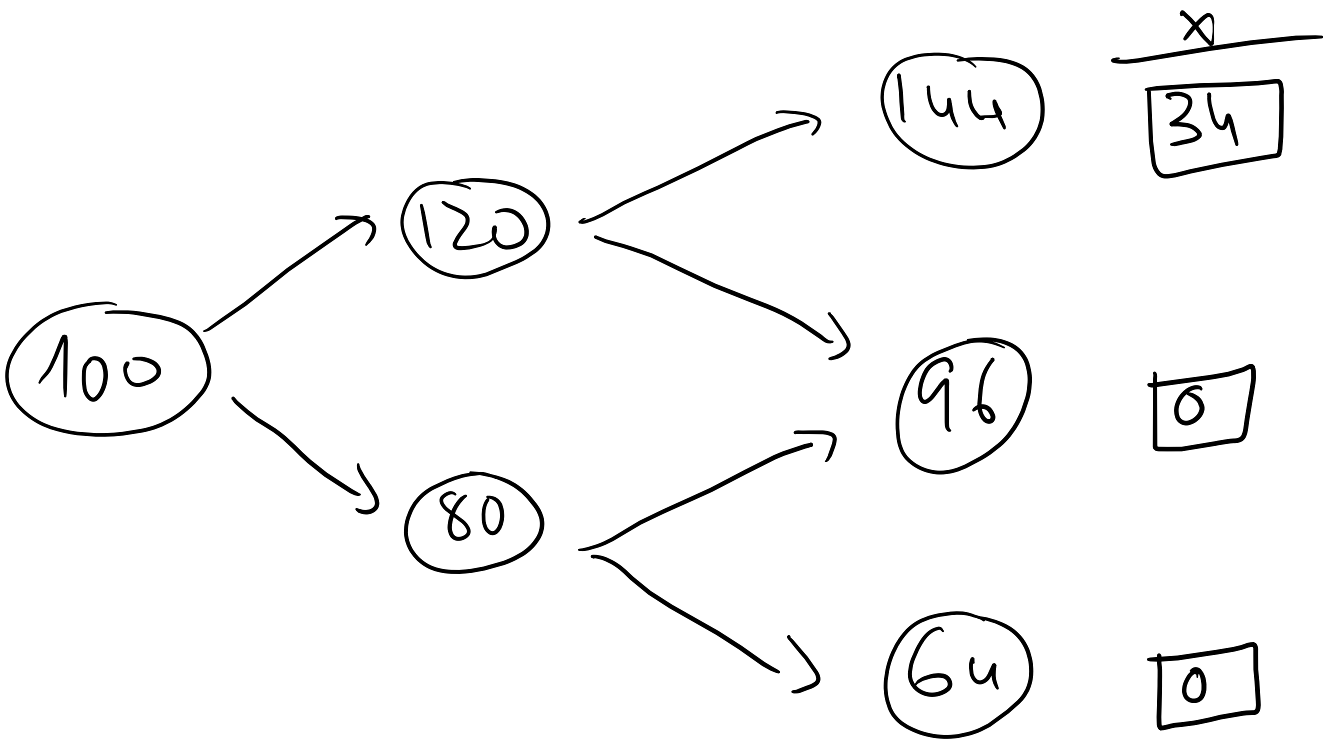
$$= 10 \times 0{,}5 + 0 \times 0{,}5 = 5$$

i.e. the price of an option X, with contract function g(St) is equal to:

$$(1+r)^{-t} E^{Q}[g(S_t)]$$

Let's see for 2 time-step?

Price of contingent claims



$$\Pi(0;X) = E^{Q}[g(S_1)] = 34 \times 0{,}5 \times 0{,}5 + 0 \times 0{,}5 \times 0{,}5 \times 2 + 0 \times 0{,}5 \times 0{,}5 = 8{,}5$$

$(34 \times q_u^2)$ $(0 \times 2 \times q_u q_d)$ $(0 \times q_d^2)$

Price of contingent claims

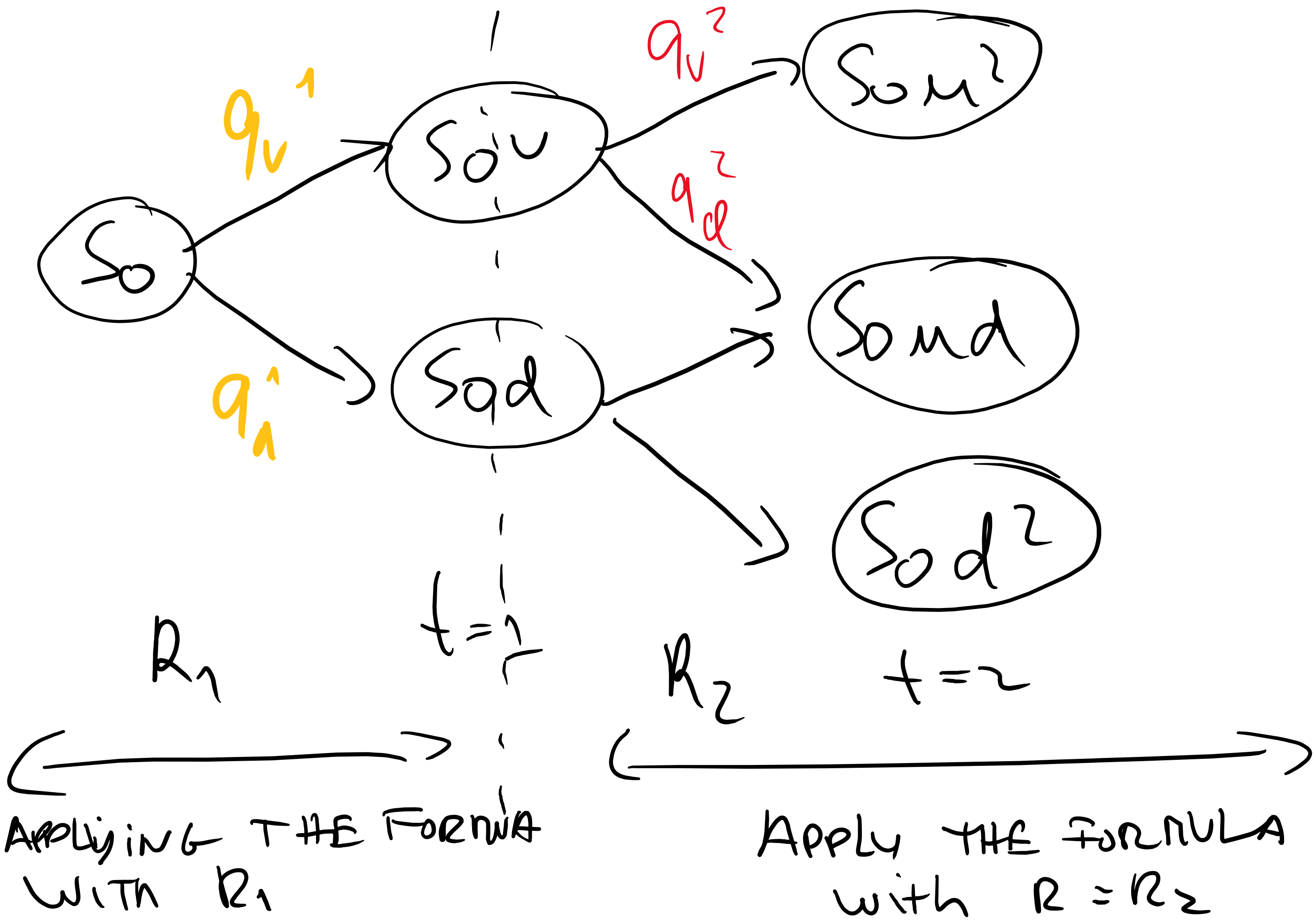
In general if we have T time steps, the price of a derivative X will be the following:

$$\pi(0;X) = (1+R)^{-T} E^{Q}[g(S_t)]$$

$$= \left(\frac{1}{1+R}\right)^{T} \sum_{k=0}^{T} g(s_0 u^k d^{T-k}) \binom{T}{k} (q_u)^k (q_d)^{T-k}$$

Price of contingent claims

In the binomial model we are assuming that r, u and d keep constant for all time steps. But we can relax that assumption.



Price of contingent claims

Fix R:
$$\Pi(0;x) = (1+R)^{-2} \sum_{k=0}^{1} g\left(s_0 u^k d^{1-k}\right)\binom{1}{k} \times$$

$$\times\, q_u^{k}\, q_d^{1-k}$$

$$q_u = \frac{1+R-d}{u-d} \quad ; \quad q_d = \frac{u-(1+r)}{u-d}$$

Price of contingent claims

With R_1 AND R_2

$$\pi(0,x) = \frac{1}{(1+R_1)}\,\frac{1}{(1+R_2)} \times \Big[g(S_0 u^2)\, q_u^1 \times q_u^2 +$$

$$+ g(S_0 u d)\,(q_u^1 \times q_d^2 + q_d^1 \times q_u^2)$$

$$+ g(S_0 d^2)\, q_d^1 \times q_d^2 \Big]$$

Price of contingent claims

RECAP:

$X \rightarrow$ contingent claim

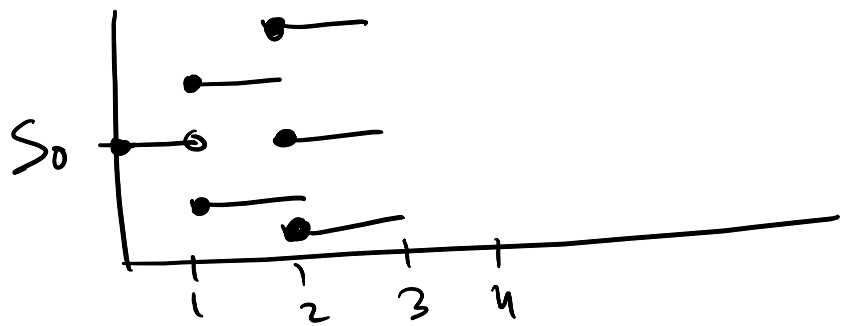
$X = g(S_T)$ where S_T is the stock price at time T

$\Pi(0; X)$: price ("prix") of the contingent claim X
at time 0

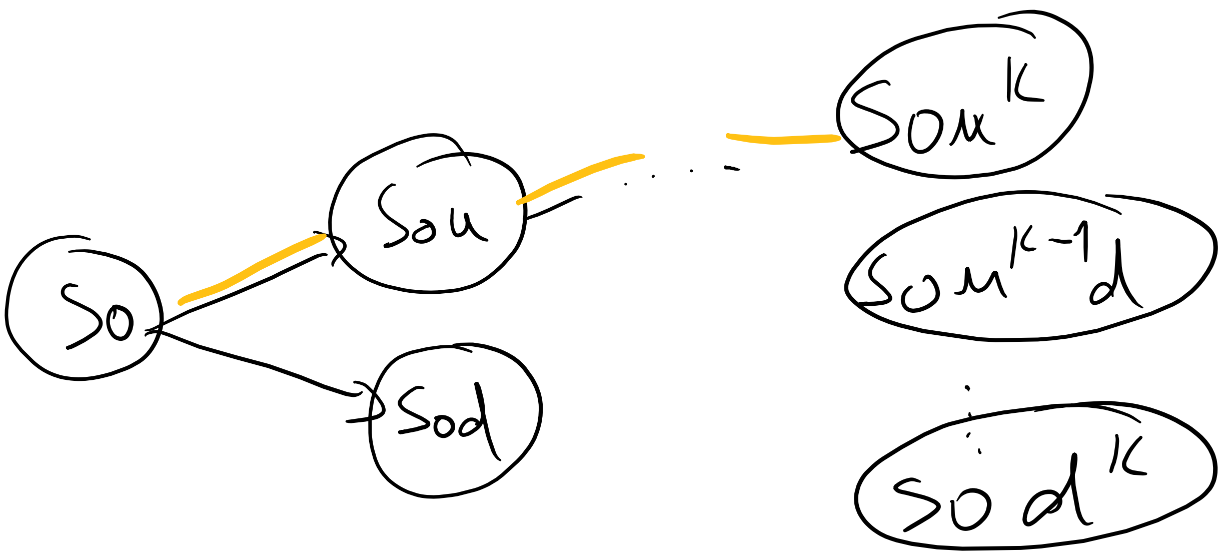
S_T : is the result of the evolution of the stock
price from time 0 to time T

Price of contingent claims

In the binomial tree we are able to "plot" all possible realizations of the stock price from time 0 to time T. Each possible realization is called a "sample path".



BINOMIAL MODEL



Price of contingent claims

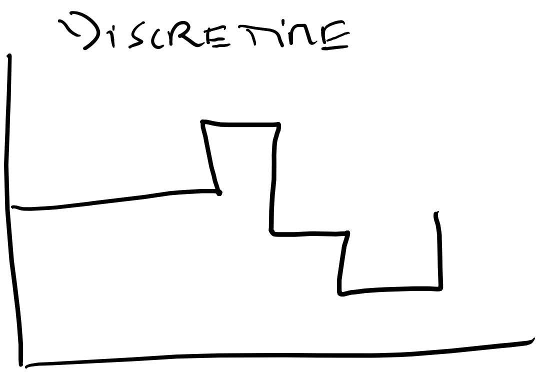
Under the binomial model we are able to price any contract. But we assume a particularly simple model:

- Discretization of the time
- The price, at each time step, is only allowed to go up by u, or go down, by d

What appends if we let:

- Time lag tends to zero?
- What is the consequence in the variations of the stock price?

In fact we want to be able to derive a model in continuous time, with continuous state space



Price of contingent claims

DISCRETE STATE SPACE

(BINOMIAL MODEL)

$$\Downarrow$$

$$\Pi(0;X) = \left(\frac{1}{1+r}\right)^{T} E^{Q}\left(g(S_T)\right)$$

RISK NEUTRAL FORMULA

CONTINUOUS STATE SPACE

(BLACK-SCHOLES MODEL)

$$\Downarrow$$

$$\Pi(0;X) = e^{-rT} E^{Q}[g(S_T)]$$

TOOLS: STOCHASTIC CALCULUS

1. exercise

Suppose that in each year the cost of a security either goes up by a factor of 2 or down by a factor of 1/2 (i.e., $u = 2$ and $d = 0.5$). Assume a continuously compounding euro interest rate of 4%.

a) If the initial price of the security is 100, determine the non-arbitrage cost of an European call option to purchase the security at the end of two periods for a strike price of 150.

b) Assume that your option is American. Should you exercise early?

2. exercise

Consider a option such that the payoff at the maturity is given by $\max(0.5(S_0 + S_T) - K, 0)$, where S_0 (S_T) denotes the stock price at initial time (maturity) and K is the strike price. Assume a binomial model, with $S_0 = 40$, $u = 1.3$, $d = 0.8$, $K = 45$ and 2 years maturity. The interest rate changes with time, in the following way: in the first year it is 3% and in the second year it is 4%.

a) Derive the price of this option.

b) Assume that the option can be exercised at the end of the first year or at maturity. When will it be optimal to exercise?

3. exercise

Suppose that c_i is the price of an European call option with maturity T and strike price K_i, with $i = 1, 2, 3$. Assume that $K_1 < K_2 < K_3$, with $K_2 = \frac{K_1+K_3}{2}$. Consider a porfolio consisting of one long position with strike price K_1, one long position with strike price K_3 and two short positions with strike price K_2, each.

a) Give the payoff at the maturity date T of this portfolio.

b) Prove that $c_2 \leq 0.5(c_1 + c_3)$.

4. exercise

A one-month European put option on a non-dividend-paying stock is currently selling for 2.5€. The stock price is 47€, the strike price is 50€, and the risk-free interest rate (monthly compounded) is 6% per annum. What opportunities are there for an arbitrageur?

5. exercise

A European call and put option on the same security both expire in three months, both have a strike price of 20 €, and both are traded for the price 3 €. If the nominal continuously compounded interest rate is 10% and the stock price is currently 25 €, identify an arbitrage.

6. exercise

Suppose that $T = 2$, and that the initial price of the stock is $S_0 = 100$. Assume that it the price may go up by a factor $u = 1.5$ or by a factor $d = 1.1$, and that the strike price, K, is equal to 150. Assume a risk-free rate of 20%. Prove that the non-arbitrage price of this call is 7.16.

7. exercise

Consider a two-year European put-option, with a strike price of 52€ on a stock whose current price is 50€. We suppose that there are two time steps of one year, and in each time step the stock price either moves up, by a proportional amount of 20%, or moves down, by a proportional amount of 20%. We also suppose that the risk-free interest rate is 5% per annum (you may assume 1-year compounded).

a) Derive the fair price of this option.

b) Now assume that this option is an American one. Describe then what is the optimal decision. In particular, when is early-exercise optimal? Derive also the value of this put.